

Colorful versions of the Lebesgue, KKM, and Hex theorem

Đorđe Baralić

Mathematical Institute SASA
Belgrade, Serbia

Rade Živaljević

Mathematical Institute SASA
Belgrade, Serbia

Abstract

Following and developing ideas of R. Karasev (Covering dimension using toric varieties, arXiv:1307.3437), we extend the Lebesgue theorem (on covers of cubes) and the Knaster-Kuratowski-Mazurkiewicz theorem (on covers of simplices) to different classes of convex polytopes (colored in the sense of M. Joswig). We also show that the n -dimensional Hex theorem admits a generalization where the n -dimensional cube is replaced by a n -colorable simple polytope. The use of specially designed *quasitoric manifolds*, with easily computable cohomology rings and the cohomological cup-length, offers a great flexibility and versatility in applying the general method.

1 Introduction

The well known connection between the classical Lyusternik–Schnirelmann category (LS-category) and the cohomological cup-length is a simple, yet elegant and powerful method of studying geometric/topological properties of a space by computable invariants arising in algebraic topology. Together with its generalizations and ramifications, this connection is indeed one of evergreen themes of geometry and topology.

It was an interesting recent observation of Karasev [9] that a similar cohomological cup-length approach can be utilized for the proof of some results of more combinatorial nature, including the following two classical results of Lebesgue, and Knaster, Kuratowski, Mazurkiewicz (KKM).

Theorem 1.1. (Lebesgue) *If the unit cube $[0, 1]^n$ is covered by a finite family $\{X_i\}_{i \in I}$ of closed sets so that no point is included in more than n sets, then one of them must intersect two opposite facets of the cube.*

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Theorem 1.2. (KKM) *If a non-degenerate simplex $\Delta^n \subset \mathbb{R}^n$ is covered by a finite family $\{F_i\}_{i \in I}$ of closed sets so that no point is covered more than n times then one of the sets F_i intersects all the facets of Δ^n .*

The method of Karasev was based on the use of cohomological properties of (both non-singular and singular) toric varieties. In particular he was able to unify Theorems 1.1 and 1.2 and interpret them as special cases of a single statement valid for all simple polytopes.

Theorem 1.3. [9, Theorem 5.2.] *Suppose that a simple polytope $P \subset \mathbb{R}^n$ is covered by a family of closed sets $\{X_i\}_{i \in I}$ with covering multiplicity at most n . Then for some $i \in I$ the set X_i intersects at least $n + 1$ facets of P .*

We continue this study by methods of toric topology, emphasizing the role of quasitoric manifolds and Davis-Januszkiewicz spaces [6, 4]. We focus on special classes of simple polytopes including the class of n -colorable simple polytopes which were introduced by Joswig in [8]. The associated classes of quasitoric manifolds have computable and often favorable cohomological properties, which have already found applications outside toric topology [2, 3].

Our central results, the ‘Colorful Lebesgue theorem’ (Theorem 3.1) and the ‘Colorful KKM-theorem’ (Theorem 4.1), together with their companions Theorem 3.2 and Theorem 4.2, are designed to include Theorems 1.1 and 1.2 as special cases and to illuminate the role of special classes of quasitoric manifolds over n -colorable and $(n + 1)$ -colorable simple polytopes.

In the same vein we prove the ‘Colorful Hex theorem’ Theorem 6.1 and describe a ‘Colorful Voronoi-Hex game’ played by n players on an n -dimensional Voronoi checkerboard.

We refer the reader, curious or intrigued by the use of the word ‘colorful’ in these statements, to [1] and [11] for a sample of results illustrating how the term ‘colorful’ gradually acquired (almost) a technical meaning in many areas of geometric and topological combinatorics.

2 Overview and preliminaries

A basic insight from the theory of *Lebesgue covering dimension* is that an n -dimensional space cannot be covered by a family \mathcal{U} of open sets which are ‘small in size’ unless we allow non-empty intersections of $(n + 1)$ sets or more. In other words if know in advance that the covering multiplicity of the family is $\leq n$, then some of the sets $U \in \mathcal{U}$ must be ‘large’ in some sense.

Theorems 1.1 and 1.2 turn this vague sense of ‘largeness’ into precise results where the combinatorics and facial structure of the cube and simplex respectively plays an important role.

Karasev [9] has found a very natural and interesting way of proving and generating such results, based on the theory of (complex and real) toric varieties. The use of the cup-length estimates is of course well known in the theory of Lusternik-Schnirelmann category and its ramifications. However, more combinatorial aspects of the problem and possibilities of the method don't seem to have been carefully explored and they certainly deserve a further study.

From this point of view it is quite natural to explore which classes of convex polytopes may provide an adequate concept of 'largeness' suitable for generalizing classical theorems of Lebesgue (on coverings of cubes) and the Knaster-Kuratowski-Mazurkiewicz theorem (on coverings of simplices).

With this goal in mind we use the theory of *quasitoric manifolds* as introduced by Davis and Januszkiewicz in the seminal paper [6] and developed by many authors, see the monograph of Buchstaber and Panov [4] (and the forthcoming, considerably updated and revised new version [5]). Quasitoric manifolds offer more flexibility and versatility than toric varieties, since they are easier to construct and their geometric and algebraic topological properties are even more closely related to combinatorics of simple polytopes.

Another input came from the theory of projectives in simplicial complexes and colorings of simple polytopes, as initiated by Joswig in [8]. In particular we focus our attention to the class of *n-colorable simple polytopes* (and some generalizations) which appear to be particularly suitable as a combinatorial framework for theorems of Lebesgue and KKM type.

2.1 Coloring of simple polytopes

An n -dimensional convex polytope P is *simple* if the number of codimension-one faces meeting at each vertex is n . Codimension-one faces are called *facets*. The following inconspicuous lemma records one of the key properties of simple polytopes.

Lemma 2.1. *If P is a simple polytope then two facets $F_1 \neq F_2$ have a non-empty intersection if and only if they share a common facet, i.e. if $F_1 \cap F_2$ is a face of P of codimension 2.*

Suppose that $\{F_i\}_{i=1}^m$ is an enumeration of all facets of a simple polytope P^n . A *proper coloring* of P^n by k colors is a map

$$h : \{F_1, \dots, F_m\} \rightarrow [k] \quad (1)$$

(or a map $h : [m] \rightarrow [k]$) such that for each two distinct facets $F_i \neq F_j$ if F_i and F_j are adjacent (in the sense that they have a common facet) then $h(F_i) \neq h(F_j)$.

In light of Lemma 2.1 it is clear that h is a coloring of a simple polytope P^n if and only if it is a coloring of the graph on $[m]$ as the set of vertices, where (i, j) is an edge if and only if $F_i \cap F_j \neq \emptyset$. For this reason the smallest number k of colors needed for a proper coloring of a simple polytope P^n is called *the chromatic number* $\chi(P^n)$.

It is immediate that $\chi(P^n) \geq n$ for any simple polytope P^n . The chromatic number of a 2-dimensional simple polytope is clearly equal to 2 or 3, depending on the parity of the number of its facets. By the *Four Color Theorem* we know that the chromatic number of a 3-dimensional polytope is either 3 or 4. However in general (for $n \geq 4$) it is far from being true that $\chi(P^n) \leq n + 1$. Actually one can easily produce simple polytopes such that their chromatic number is exactly the number of their facets. Examples include polytopes which arise as polars of cyclic polytopes $C^n(m)$, see [4, Example 0.6, p.11].

In spite of that the class of n -colorable n -dimensional simple polytopes is quite large, with many interesting examples. It is known that this class is closed for products [4, Construction 1.12, p.10] and connected sums [4, Construction 1.13, p.10]. From any given simple polytope P^n by truncation over all its faces we obtain a simple polytope Q^n such that $\chi(Q^n) = n$. The complete description of this class is given by M. Joswig in [8], who proved that a simple n -polytope P^n admits an n -coloring if and only if every 2-face has an even number of edges. For this reason an n -colorable polytope is sometimes referred to as *Joswig polytope*.

Definition 2.1. Suppose that P^n is an n -colorable simple polytope and let h be an associated coloring function (1). For $0 \leq k \leq n$ let $I = \{i_1, i_2, \dots, i_{n-k}\} \subset [n]$ be a collection of $(n-k)$ colors. We say that a k -dimensional face K of P^n is in the I -color class if $I = \{h(F) \mid K \subset F\}$.

Example 2.1. The n -dimensional cube $I^n \subset \mathbb{R}^n$ is an n -colorable simple polytope with colors corresponding to axes of symmetry of pairs of opposite facets (coordinate axes). Similarly, I -color classes of k -faces correspond to $(n-k)$ -dimensional coordinate subspaces of the ambient space \mathbb{R}^n .

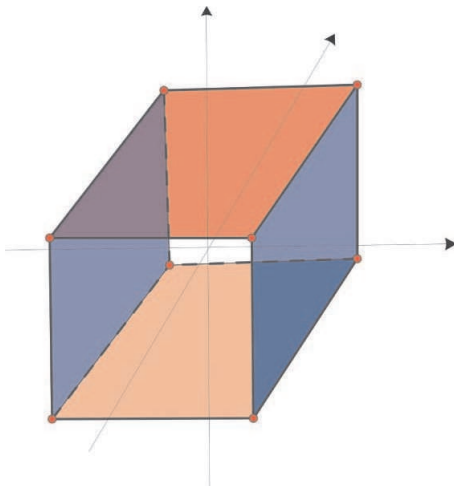


Figure 1: Coloring of the cube I^3 (the case $n = 3$).

2.2 Preliminaries on quasitoric manifolds

A quasitoric manifold (originally a "toric manifold") is a topological counterpart to the nonsingular projective toric variety (of algebraic geometry). A smooth $2n$ -dimensional manifold M^{2n} is a quasitoric manifold if it admits a smooth, locally standard action of an n -dimensional topological torus $T^n = (S^1)^n$, with an n -dimensional simple convex polytope P^n as the orbit space. Quasitoric manifolds were introduced by Davis and Januszkiewicz in [6] and developed by many authors, see the monograph of Buchstaber and Panov [4] and the forthcoming new version [5] summarizing the development of the theory in the last two decades.

The facets F_j of the polytope P^n correspond to T^{n-1} -orbits and the associated stabilizer groups define the *characteristic map* (characteristic matrix) $\lambda : F_j \mapsto T(F_j)$, where $T(F_j) = (\lambda_{ij})_{i=1}^n \in \mathbb{Z}^n$ is a unimodular vector.

Conversely, each $n \times m$ *characteristic matrix* $\lambda = (\lambda_{ij})$ produces a $2n$ -dimensional quasitoric variety M^{2n} over a simple n -dimensional polytope P^n , provided the column vectors $\lambda_j = (\lambda_{ij})$ satisfy the condition that $\lambda_{j_1}, \dots, \lambda_{j_n}$ is a \mathbb{Z}^n -basis for each choice of facets F_{j_1}, \dots, F_{j_n} having a common vertex.

Following Davis-Januszkiewicz [6, Theorem 4.14.] there is an isomorphism,

$$H^*(M^{2n}; \mathbb{Z}) \cong \mathbb{Z}[v_1, \dots, v_m] / \langle I + J \rangle \quad (2)$$

where I is the *Stanley-Reisner* ideal of P (generated by monomials $v_{i_1} \dots v_{i_k}$ such that $F_{i_1} \cap \dots \cap F_{i_k} = \emptyset$) and J is the ideal generated by linear forms which corresponds to the rows of the characteristic matrix (λ_{ij}) .

2.3 Lyusternik-Schnirelmann method

Definition 2.2. For a given cohomology class $\omega \in H^*(X)$ we say that a closed (open) subset $F \subset X$ is ω -inessential (or simply inessential if ω is clear from the context) if ω is mapped to zero by the restriction map,

$$H^*(X) \longrightarrow H^*(F).$$

The following well known ‘lemma’ captures the essence of the Lyusternik-Schnirelmann method.

Lemma 2.2. Assume that $\{X_i\}_{i=1}^n$ is a collection of closed (open) subsets of a space X and let $\{\omega_i\}_{i=1}^n$ be a collection of cohomology classes in $H^*(X)$. If X_i is ω_i -inessential for each $i = 1, \dots, n$ then $Z = \bigcup_{i=1}^n X_i$ is ω -inessential where $\omega = \omega_1 \dots \omega_n$.

The following proposition [9, Lemma 3.2.] is the key result connecting the *covering multiplicity* of a finite family $\{Y_i\}_{i \in I}$ of subspaces of Y with the cup-length of the ring $H^*(Y)$. Recall that the covering multiplicity of $\{Y_i\}_{i \in I}$ is $\leq k$ if for each $y \in Y$ the cardinality of the set $\{i \in I \mid y \in Y_i\}$ is at most k .

Proposition 2.1. *Suppose that a finite family $\mathcal{U} = \{U_i\}_{i=1}^N$ of open subsets in a paracompact space Y has covering multiplicity at most m . Assume that for some $\omega \in H^*(Y)$ each of the sets U_i is ω -inessential. Then the union $\cup_{i=1}^N U_i$ is ω^m -inessential.*

Corollary 2.1. *Assuming that the cohomology theory satisfies a suitable continuity condition (as the Alexandrov-Čech theory) the Proposition 2.1 is valid for finite closed coverings of multiplicity $\leq m$.*

3 Colorful Lebesgue theorem

Suppose that P^n is an n -colorable simple polytope (Section 2.1) with m facets F_1, \dots, F_m and the corresponding coloring function (1). Let e_1, \dots, e_n be the standard basis of the lattice \mathbb{Z}^n .

Definition 3.1. *The coloring (1) gives rise to a canonical characteristic function λ where $\lambda(F_i) = e_{h(i)}$. The quasitoric manifold arising from this construction is referred to as the canonical quasitoric manifold of the pair (P^n, h) or simply as a canonical quasitoric manifold associated to the n -colorable simple polytope P^n .*

Suppose that M^{2n} is the canonical quasitoric manifold associated to an n -colorable simple polytope P^n . Let $\pi : M^{2n} \rightarrow P^n$ be the corresponding projection map. For each facet F_i the set $M_i := \pi^{-1}(F_i)$ is a codimension 2 submanifold of M^{2n} . Let $v_i \in H^2(M^{2n}; \mathbb{Z})$ be the Poincaré dual of the fundamental class $[M_i] \in H_2(M^{2n}; \mathbb{Z})$ (relative to some (omni)orientation on M^{2n}).

According to the Davis-Januszkiewicz description of the cohomological ring of M^{2n} [6, Theorem 4.14.] there is an isomorphism,

$$H^*(M^{2n}; \mathbb{Z}) \cong \mathbb{Z}[v_1, \dots, v_m] / \langle I + J \rangle \quad (3)$$

where I is the *Stanley-Reisner* ideal of P (generated by monomials $v_{i_1} \dots v_{i_k}$ such that $F_{i_1} \cap \dots \cap F_{i_k} = \emptyset$) and J is the ideal generated by linear forms L_i where,

$$L_i(v_1, \dots, v_m) = \sum_{h(j)=i} v_j. \quad (4)$$

The following proposition records some of the properties of the cohomology ring of the canonical quasitoric manifold M^{2n} associated to an n -colorable simple polytope P^n (Definition 3.1).

Proposition 3.1.

- (1) *The product $v_i v_j$ of two distinct classes of the same color is zero in $H^*(M^{2n}; \mathbb{Z})$.*
- (2) *The sum of all ‘classes of the same color’ vanishes, $\sum_{h(i)=k} v_i = 0$.*
- (3) *The square v_i^2 of any generator v_i is zero in $H^*(M^{2n}; \mathbb{Z})$.*

- (4) Suppose that $\{F_{i_k}\}_{k=1}^n$ are all facets which share a common vertex V of P^{2n} . Then, $(v_{i_1} + \dots + v_{i_n})^n$ is a non-zero class in $H^{2n}(M^{2n}; \mathbb{Z})$.

Proof. The first observation is a direct consequence of the fact that $v_i v_j \in I$ if $h(F_i) = h(F_j)$ and $i \neq j$. The second property is just a restatement of the equation (4) describing the ideal J . The property (3) follows on multiplying the both sides of the equation $L_i = 0$ by v_i . Finally (4) follows from the observation that $v_{i_1} \dots v_{i_n}$ is the fundamental cohomology class in $H^{2n}(M^{2n}; \mathbb{Z})$ and the equality,

$$(v_{i_1} + \dots + v_{i_n})^n = n! v_{i_1} \dots v_{i_n}.$$

□

The following result extends the Lebesgue theorem (Theorem 1.1) to the class of n -colorable simple polytopes.

Theorem 3.1. (Colorful Lebesgue theorem) *Suppose that an n -colorable simple polytope P^n is covered by a family of closed sets $P^n = \cup_{i=1}^N X_i$ such that each point $x \in P^n$ is covered by no more than n of the sets X_j . Then for some i , a connected component of X_i intersects at least two distinct facets of P^n of the same color.*

Proof. Without loss of generality we may assume that all sets X_i are connected. Indeed, the connected components of all sets X_j define a covering of P^d which also satisfies the conditions of the theorem. Let M^{2n} be the canonical quasitoric manifold over P^n (Definition 3.1) and let $\pi : M^{2n} \rightarrow P^n$ be the associated projection map.

Assume (for contradiction) that each of the sets X_j intersects at most one facet of each of the colors $i \in [n]$. Given a vertex V of P , let $\{F_{i_k}\}_{k=1}^n$ be the collection of all facets of P^n incident to V where $h(F_{i_k}) = h(i_k) = k$ for the chosen coloring function (1). By assumption for each k either $F_{i_k} \cap X_j = \emptyset$ (and $\pi^{-1}(X_j)$ is automatically v_{i_k} -inessential) or $F_i \cap X_j = \emptyset$ for each $F_i \neq F_{i_k}$ in the chosen color class ($h(i) = k$). In the latter case $\pi^{-1}(X_j)$ is v_i -inessential for each i such that $h(i) = k$ and $F_i \neq F_{i_k}$. Since the sum of all classes of the same color vanishes (Proposition 3.1) we conclude that $\pi^{-1}(X_j)$ is v_{i_k} -inessential in this case as well.

Summarizing, we observe that $\pi^{-1}(X_j)$ is ω -inessential for each j where $\omega = v_{i_1} + \dots + v_{i_n}$. It follows from Proposition 2.1 (Corollary 2.1) that $M^{2n} = \cup_{j=1}^N \pi^{-1}(X_j)$ is ω^n -inessential which is in contradiction with Proposition 3.1. □

Theorem 3.1 extends the Lebesgue theorem (Theorem 1.1) to the class of all n -colorable simple polytopes. Informally it says that if a collection $\{X_i\}_{i=1}^N$ of closed subsets of P^n has “small multiplicity” (multiplicity $\leq n$) and sets of “small diameter” ($X_i \cap F_j \neq \emptyset$ for at most one index j) then it cannot be a covering of P^n .

Karasev proved [9, Theorem 4.2.] a very interesting extension of Theorem 1.1 where he was able to show that in a very precise sense the smaller is the multiplicity of $\{X_i\}_{i=1}^N$ the bigger are the connected components of $P^n \setminus \cup_{i=1}^N X_i$. He obtained this result by applying his method to the toric variety $(\mathbb{C}P^1)^n$ over the cube I^n . Our objective is to extend this result to the class of n -colorable simple polytopes.

A ‘vertex class’ $\omega \in H^2(M^{2n})$, associated to a vertex $V \in P^n$, is by definition the sum $\omega = v_1 + \dots + v_n$ of all 2-classes dual to facets F_i incident to V .

Theorem 3.2. *Suppose that P^n is an n -colorable simple polytope, M^{2n} its canonical quasitoric manifold, and $\pi : M^{2n} \rightarrow P^n$ the associated projection map. Let $\omega = v_1 + \dots + v_n$ be the 2-dimensional ‘vertex class’ associated to a vertex $V \in P^n$. Suppose that $\mathcal{F} = \{X\}_{i=1}^N$ is a finite family of closed subsets of P^n such that each X_i intersects at most one of the facets in each of the color classes. If the covering multiplicity of \mathcal{F} is at most $k \leq n$ then there exists a connected component Z of the set $P^n \setminus \cup_{i=1}^N X_i$ which is ω^{n-k} -essential in the sense that the restriction of the class ω^{n-k} on $\pi^{-1}(Z)$ is non-trivial. Moreover, if \mathcal{K} is the collection of all k -faces K of P^n such that $Z \cap K \neq \emptyset$ then \mathcal{K} contains a collection of k -faces of size at least 2^{n-k} which are all in the same I -color class for some $I = \{i_1, \dots, i_{n-k}\} \subset [n]$ (Definition 2.1).*

Proof. For a chosen vertex $V \in P^n$ let $\{F_i\}_{i=1}^n$ be the collection of all facets incident to V (we assume that $h(F_i) = i$). If $v_i \in H^2(M^{2n}; \mathbb{Z})$ is the class associated to the facet F_i then (following Proposition 3.1) the class ω^n is non-zero where $\omega = v_1 + \dots + v_n$. Moreover we observe that,

$$\omega^k = k! \sum_J v_J \neq 0 \quad (5)$$

where the sum is taken over all collections $J = \{j_1, \dots, j_k\}$ of colors of size k and $v_J = v_{j_1} v_{j_2} \dots v_{j_k}$.

Simplifying the notation, from here on we say that $Y \subset P^n$ is ω^k -inessential if the set $\pi^{-1}(Y) \subset M^{2n}$ is ω^k -inessential. Assuming that each X_i intersects at most one of the facets in each of the color classes we deduce (as in the proof of Theorem 3.1) that the set $\cup_{i=1}^N X_i$ and is ω^k -inessential. Moreover (assuming the cohomology is continuous) this holds also for some small open neighborhood U of $\cup_{i=1}^N X_i$. It follows that the restriction of ω^{n-k} on $\pi^{-1}(W)$ is non-trivial where $W = P^n \setminus \cup_{i=1}^N X_i$. Otherwise W would be ω^{n-k} -inessential and $P^n = U \cup W$ would be ω^n -inessential (contradicting Proposition 3.1). Since W is ω^{n-k} -essential the same holds for some connected component Z of W .

In order to prove the second part of the theorem it will be sufficient to show that there exists a collection of monomials $v_J = v_{j_1} v_{j_2} \dots v_{j_{n-k}}$ of size $\geq 2^{n-k}$ such that Z is v_J -essential and all these monomials are in the same I -color class (in the sense of Definition 2.1). Indeed, by the same argument as before, if Z is v_J -essential then $Z \cap K \neq \emptyset$ where $K = F_{j_1} \cap \dots \cap F_{j_{n-k}}$.

In light of the fact that Z is ω^{n-k} -essential, by inserting $n - k$ in the place of k in the equality (5) we observe that at least one of the monomials $v_J = v_{j_1} v_{j_2} \dots v_{j_{n-k}}$ must be non-zero in $H^*(Z)$. Since (Proposition 3.1),

$$v_{j_1} = \sum \{v_j \mid j \neq j_1 \text{ and } h(j) = h(j_1)\} \quad (6)$$

we can replace the generator v_{j_1} in the monomial v_J by the sum of the remaining generators v_j of the same ‘color’ ($h(j) = h(j_1)$). In other words we multiply both sides of the equality (6) by the monomial $v_{j_2} \dots v_{j_{n-k}}$ and observe that on the right hand side there must appear a monomial $v'_J = v_j v_{j_2} \dots v_{j_{n-k}}$, in the same I -color class as v_J , which is also non-zero in $H^*(Z)$. This procedure can be continued for other indices (generators) which guarantees that there exist at least 2^{n-k} different monomials in the same I -color class which are all non-zero in $H^*(Z)$. \square

Remark 1. The proof of Theorem 3.2 shows that a vertex V of P^n and the associated vertex class $\omega = v_1 + \dots + v_n$ can be prescribed in advance. From here we deduce that the vertex V is certainly contained by one of the 2^{n-k} k -dimensional faces in the same I -color class which intersect Z .

4 Colorful KKM-theorem

In this section we prove a colorful version of Knaster-Kuratowski-Mazurkiewicz ‘lemma’ (Theorem 1.2). The strategy is the same as in the previous section. We describe a family of simple polytopes together with associated natural quasitoric manifolds and show that each of them has a special cohomology class ω such that $\omega^n \neq 0$.

Definition 4.1. Suppose that a simple polytope P^n can be colored by $(n+1)$ colors (in the sense of Section 2.1) and for a chosen coloring let $\{T_1, \dots, T_k\} = h^{-1}(n+1)$ be the collection of all facets colored by the color $n+1$. The polytope P^n is called specially $(n+1)$ -colorable if the associated coloring function $h : \{F_1, \dots, F_m\} \rightarrow [n+1]$ has the property that all facets $\{T_i\}_{i=1}^k$ are n -simplices.

An immediate example of a special $(n+1)$ -colorable polytope is the standard simplex Δ^n . A large class of such polytopes is obtained by truncating n -colorable polytopes at an odd number of (strongly separated) vertices (Figure 2).

In the following definition we introduce a class of *canonical quasitoric manifolds* associated to a $(n+1)$ -colorable simple polytope with a distinguished color (the color $n+1$).

Definition 4.2. Suppose that P^n is a $(n+1)$ -colorable, simple polytope. For some enumeration $\{F_1, \dots, F_m\}$ of its facets let $h : [m] \rightarrow [n+1]$ be a chosen coloring function. Let e_1, \dots, e_n be the standard basis in \mathbb{Z}^n and let $e_\epsilon = \epsilon_1 e_1 + \dots + \epsilon_n e_n$ where $\epsilon_i \in \{-1, +1\}$. Define the characteristic function $\lambda_\epsilon : \{F_1, \dots, F_m\} \rightarrow \mathbb{Z}^n$ by the equation,

$$\lambda_\epsilon(F_i) = \begin{cases} e_{h(i)} & \text{if } i \neq n+1 \\ e_\epsilon & \text{if } i = n+1 \end{cases} \quad (7)$$

The quasitoric manifold M_ϵ^{2n} associated to the pair (P^n, λ_ϵ) is called the canonical quasitoric manifold of the $(n+1)$ -colored polytope P^n with the distinguished color $n+1$ and the defining sign vector e_ϵ .

By definition there are 2^n distinct canonical quasitoric manifolds M_ϵ^{2n} associated to a $(n+1)$ -colored polytope P^n . From here on we select $e = -e_1 - \dots - e_n$ as the preferred sign vector and denote the corresponding manifold by M^{2n} .

Proposition 4.1 collects some of the properties of the cohomology ring $H^*(M^{2n}; \mathbb{Z})$. This ring is, in agreement with (3), a quotient ring of $\mathbb{Z}[v_1, \dots, v_m]$ where v_i is the 2-dimensional cohomology class associated to the facet F_i . For bookkeeping purposes we modify (refine) this notation as recorded by the following definition.

Definition 4.3. Following the notation of Definition 4.1, let t_i be the variable associated to the facet T_i . If $\{F_{i\nu}\}_{\nu \in S_i}$ are facets colored by the color $i \in [n]$ then the associated 2-classes are $v_{i\nu}$. By assumption T_j is a simplex for each $j = 1, \dots, k$. It follows that for each color $i \in [n]$ there is a unique facet $F_{ij} := F_{i\nu_j}$ adjacent to T_j . The associated dual cohomology class is denoted by v_{ij} .

Proposition 4.1. Suppose that P^n is a specially $(n + 1)$ -colorable polytope (Definition 4.1) and let M^{2n} be the associated canonical quasitoric manifold corresponding the sign vector $e = -e_1 - \dots - e_n$ (Definition 4.2). Then the cohomology ring $H^*(M^{2n}; \mathbb{Z})$, as described by (3), has the following properties.

- (a) The Stanley-Reisner ideal I of P^n contains all monomials $v_i v_j$ such that $h(i) = h(j)$ and $i \neq j$. In particular $t_i t_j \in I$ and $v_{i\nu_1} v_{i\nu_2} \in I$ for $i \neq j$ and $\nu_1 \neq \nu_2$.
- (b) For each $i \in [n]$ there is a linear relation in the cohomology ring $H^*(M^{2n}; \mathbb{Z})$,

$$t_1 + \dots + t_k = \sum_{\nu \in S_i} v_{i\nu} \quad (8)$$

- (c) For each $j = 1, \dots, k$ there is a relation (Definition 4.3)

$$t_j^2 = t_j v_{ij}.$$

Proof. The proof follows the same pattern as the proof of Proposition 3.1. For example on multiplying the relation (8) by t_j one obtains $t_j^2 = t_j v_{i\nu_j}$. Note that the equality (8) is a consequence of our choice of $e = -e_1 - \dots - e_n$ as the preferred sign vector in the definition of the canonical quasitoric manifold (Definition 4.2). \square

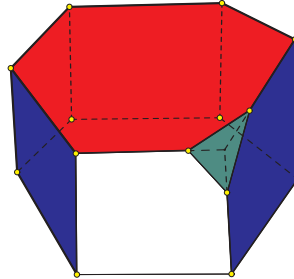


Figure 2: A truncated $(n + 1)$ -colorable polytope (the case $n = 3$).

Proposition 4.2. The class t_j^n is equal to the fundamental cohomology class of M^{2n} for each $j = 1, \dots, k$.

Proof. It follows from Proposition 4.1 that,

$$t_j^n = t_j^2 t_j^{n-2} = v_{1j} t_j^{n-1} = v_{1j} v_{2j} t_j^{n-2} = \dots = v_{1j} v_{2j} \dots v_{n-1,j} t_j. \quad (9)$$

If $\pi : M^{2n} \rightarrow P^n$ is the projection map then t_j is the dual to the fundamental homology class of the (properly oriented) manifold $\pi^{-1}(T_j)$, similarly v_{ij} is dual to the fundamental homology class of $\pi^{-1}(F_{ij})$. All these manifolds intersect transversely and their intersection is a single point. It follows from Poincaré duality that $v_{1j}v_{2j} \dots v_{n-1,j}t_j$ is the dual of a generator of $H_0(M^{2n}; \mathbb{Z})$ so both $v_{1j}v_{2j} \dots v_{n-1,j}t_j$ and t_j^n are fundamental classes of M^{2n} . \square

Theorem 4.1. (Colorful KKM theorem) *Let P^n be a specially $(n+1)$ -colorable polytope in the sense of Definition 4.1. Suppose that P^n is covered by a family of closed sets $P^n = \cup_{i=1}^N X_i$ with the covering multiplicity $\leq n$ (i.e. each point $x \in P^n$ is covered by no more than n of the sets X_j). Then there exists $i \in [N]$ and a connected component Y_i of X_i such that among the faces of P^n intersected by Y_i are facets of all $n+1$ colors.*

Proof. As in the proof of Theorem 3.1 we are allowed to assume that all sets X_i are connected. Let M^{2n} be the canonical quasitoric manifold associated to P^n corresponding to the sign vector $e = -e_1 - \dots - e_n$ (Definition 4.2 and Proposition 4.1). Let $t = t_1 + t_2 + \dots + t_k$ be the sum of all 2-classes corresponding to simplicial facets T_j of the polytope P^n .

If $X_j \cap F_{i\nu} = \emptyset$ for each facet $F_{i\nu}$ of color ν then X_j is $v_{i\nu}$ -inessential for each $\nu \in S_i$. We deduce from Proposition 4.1 (b) that X_j is t -inessential as well and by Proposition 2.1 (Corollary 2.1) we know that $M^{2n} = \cup_{j=1}^N \pi^{-1}(X_j)$ is t^n -inessential. This contradicts the fact that the class (Proposition 4.2)

$$t^n = (t_1 + \dots + t_k)^n = t_1^n + \dots + t_k^n = kt_1^n$$

is a non-trivial element of $H^{2n}(M^{2n}; \mathbb{Z})$. \square

It is certainly possible to refine Theorem 4.1 along the lines of Theorems 2.1. and 4.2. in [9] and our Theorem 3.2. The following corollary of the proof of Theorem 4.1 summarizes the cohomological content of such a result.

Theorem 4.2. *Let P^n be a specially $(n+1)$ -colorable polytope (Definition 4.1). Suppose that P^n is covered by a family of closed sets $P^n = \cup_{i=1}^N X_i$ with the covering multiplicity $k \leq n$ and there is no X_i intersecting some $n+1$ distinct colored facets. Then there exists a connected component W of $P^n \setminus \cup_{i=1}^N X_i$ which is t^{n-k} -essential in the sense that the restriction of the class t^{n-k} on $\pi^{-1}(W) \subset M^{2n}$ is non-trivial. Moreover, if \mathcal{K} is the collection of all k -faces K of P^n such that $W \cap K \neq \emptyset$ then \mathcal{K} contains a k -skeleton of some simplicial face T_i and at least $\binom{n}{k}$ k -faces of P^n not contained in T_i .*

Proof. The proof uses the same arguments as the first half of the proof of Theorem 3.2 so we omit the details. \square

5 General Polytopes

In this section we briefly address the case of general (not necessarily simple) polytopes. We use the fact that after truncations along all the faces of a polytope P^n we obtain a

Joswig polytope \overline{P}^n . Indeed, facets F_K of \overline{P}^n are naturally indexed by faces K of P^n and a proper coloring of \overline{P}^n by n colors is defined by $h(F_K) = \dim(K)$.

Theorem 5.1. *Let a polytope P^n be covered by a family of closed sets $\{X_i\}_{i=1}^N$ with covering multiplicity at most n . Then some connected component of X_i intersects at least two different k -faces of P^n (for some k).*

Proof. Let \overline{P}^n be the total truncation of P^n such that $\partial\overline{P}^n$ lies in ε neighborhood of ∂P^n , where $\varepsilon > 0$ is a sufficiently small, positive number. Observe that the restriction of the family $\{X_i\}_{i=1}^N$ to \overline{P}^n is a covering of \overline{P}^n by closed subsets. Theorem 3.1 implies that some connected component of X_i must intersect at least two facets F_{K_1} and F_{K_2} of \overline{P}^n , corresponding to k -faces K_1 and K_2 of P^n . The result follows by a limiting argument (when $\varepsilon \rightarrow 0$). \square

6 Colorful Hex Theorem

As illustrated in previous sections quasitoric manifolds are a very useful tool for analyzing various generalizations of the Lebesgue and KKM theorem. Karasev observed in [9, Theorem 4.3.] that the Lyusternik-Schnirelmann method is equally useful for the proof of the n -dimensional *Hex theorem* [7].

Suppose that $\{A_{i1}, A_{i2}\}_{i=1}^n$ is some labelling of pairs of opposite facets of the n -dimensional cube I^n . The Hex Theorem claims that for each covering $I^n = \cup_{i=1}^n X_i$ of the n -cube I^n by closed sets there exists an index i and a connected component Y_i of X_i such that both $Y_i \cap A_{i1} \neq \emptyset$ and $Y_i \cap A_{i2} \neq \emptyset$.

At first site this result is an immediate consequence and a very special case of Theorem 1.1, however on closer inspection we see that this is not the case. Indeed in the Hex Theorem we can prescribe in advance a matching between closed sets X_i and the corresponding pairs $\{A_{i1}, A_{i2}\}$ of opposite facets of I^n .

For this reason we formulate and prove a Hex analogue of Theorem 3.1 which contains [9, Theorem 4.3.] as a special case.

Theorem 6.1. (Colorful Hex theorem) *Suppose that P^n is an n -colorable simple polytope and let $h : [m] \rightarrow [n]$ be a selected coloring function which associates to each facet F_j the corresponding color $h(F_j) = h(j)$. Let V be a vertex of P^n and let $\{F_{\nu_i}\}_{i=1}^n$ be the collection of all facets of P^n which contain V such that $h(\nu_i) = i$.*

Suppose that the polytope P^n is covered by a family of n closed sets $P^n = \cup_{i=1}^n X_i$. Then for some i , a connected component of X_i intersects both F_{ν_i} and some other facet F_j of P^n colored by the color i .

Proof. Let M^{2n} be the canonical quasitoric manifold associated to the simple n -colorable polytope P^n with the chosen coloring $h : [m] \rightarrow [n]$ and let $\pi : M^{2n} \rightarrow P^n$ be the corresponding projection map. Let v_j be the class associated to the facet F_j , in particular v_{ν_i} is the class associated to the facet F_{ν_i} .

It is sufficient to show that there exists a color i such that X_i is v_{ν_i} -essential in the sense that the restriction of the class v_{ν_i} on $\pi^{-1}(X_i)$ is non-zero. Indeed, in this case some connected component Y_i of X_i would be also v_{ν_i} -essential. From here it would follow that for some $v_j \neq v_{\nu_i}$ such that $h(j) = i$ the set Y_i would be v_i -essential as well. Otherwise, in light of the relation,

$$\sum_{h(j)=i} v_j = 0$$

we would deduce that Y_i is NOT v_{ν_i} -essential. This observation would complete the proof since Y_i would certainly, under these condition, have a non-empty intersection with both F_{ν_i} and F_i .

Let us assume now that X_i is v_{ν_i} -inessential for each $i = 1, \dots, n$. It follows from the Lyusternik-Schnirelmann Lemma 2.2 that $M^{2n} = \cup_{i=1}^n \pi^{-1}(X_i)$ is ω -inessential where $\omega = v_{\nu_1} v_{\nu_2} \cdots v_{\nu_n}$. This contradicts the fact that ω is the fundamental cohomology class of M^{2n} . \square

6.1 A generalized Game of Hex

According to Wikipedia, http://en.wikipedia.org/wiki/Hex_%28board_game%29, see also [7, 10], it was *John Nash* who originally proved that the game of Hex cannot end in a tie. Moreover, he is attributed to be the first who observed that the first player has the winning strategy for the game of Hex on the usual (rhombic) game board.

Here, as an application of Theorem 6.1, we describe a fairly general version of the Game of Hex [7] played by n players which also cannot end in an undecided position.

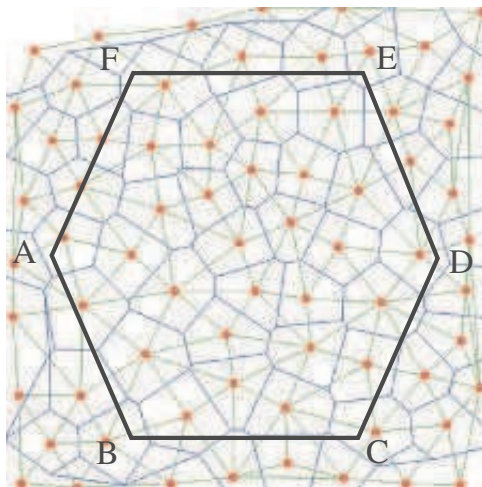


Figure 3: A hexagonal Voronoi checkerboard.

Let $S \subset \mathbb{R}^n$ be a finite set of points and let $\{V_x\}_{x \in S}$ the associated Voronoi partition of \mathbb{R}^n . Let P^n be a simple n -colorable polytope with m facets and an associated coloring

function $h : [m] \rightarrow [n]$. Choose a vertex V of P^n and let $\{F_{\nu_i}\}_{i=1}^n$ be the collection of all facets containing the vertex V such that $h(\nu_i) = i$.

There are n players J_1, \dots, J_n (each player J_i is assigned the corresponding color i). The first player chooses a point $x_1 \in S$ and colors the corresponding Voronoi cell V_{x_1} by the color 1. The second player chooses a point $x_2 \in S \setminus \{x_1\}$ and colors the Voronoi cell V_{x_2} by color 2, etc. After the first round of the game the first player chooses a point $x_{n+1} \in S \setminus \{x_1, \dots, x_n\}$, etc.

The game continues until one of the players (say the player J_i) creates a connected monochromatic set of cells (all of color i) which connect the facet F_{ν_i} with one of the facets F_j such that $h(j) = i$. Alternatively the game ends if there are no more points in S to distribute among players.

An easy application of Theorem 6.1 shows that the game will always be decided i.e. sooner or later one of the players will win the game.

Figure 3 illustrates the simplest new case of the game. The red player chooses edges AB, CD, EF of the hexagon, while remaining edges belong to the blue player. The red player tries to connect the edge AB with either the edge CD or EF , similarly the blue player tries to connect the edge AF with either CD or EF . As predicted by the Colorful Hex Theorem sooner or later one of the players will achieve her goal.

References

- [1] J.L. Arocha, I. Bárány, J. Bracho, R. Fabila, L. Montejano. Very colorful theorems. *Discrete Comput. Geom.* 42, 142–154, (2009).
- [2] Dj. Baralić. Immersions and embeddings of quasitoric manifolds over cube. *Publications de l'Institut Mathématique* (2014) (N.S.) 95 (109), 63–71.
- [3] Dj. Baralić and V. Grujić. Immersions and Embeddings of Small Covers and Quasitoric Manifolds over n -Colored Simple Polytopes.
- [4] V. Buchstaber and T. Panov. *Torus Actions and their applications in topology and combinatorics*, AMS University Lecture Series, volume 24, (2002).
- [5] V. Buchstaber and T. Panov. *Toric Topology*, arXiv:1210.2368v3 [math.AT].
- [6] M. Davis and T. Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. *Duke Math. J.* **62** (1991), no. 2, 417–451.
- [7] D. Gale. The game of hex and the Brouwer fixed-point theorem. *Amer. Math. Monthly* 86 (1979), 818–827.
- [8] M. Joswig. Projectivities in simplicial complexes and colorings of simple polytopes. *Math. Z.* (2002) **240**:243–259.
- [9] R. Karasev. Covering dimension using toric varieties, arXiv:1307.3437

- [10] J. Matoušek, G. M. Ziegler, A. Björner. Around Brouwer's fixed point theorem (Lecture notes). arXiv:1409.7890 [math.CO].
- [11] G. Ziegler. $3N$ Colored points in a plane. *Notices of the AMS* (2011), Vol. 54, No. 4, 550–557